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Boundary Conditions for Euler Equations

Sylvie Benzoni-Gavage* and Jean-François Coulombel†
Centre National de la Recherche Scientifique, 69364 Lyon Cedex 07, France
and
Stéphane Aubert‡
Ecole Centrale de Lyon, 69134 Ecully Cedex, France

The mathematical theory of boundary conditions is applied to Euler equations of gasdynamics to select appropriate boundary conditions for numerical experiments. We begin with the presentation of the mathematical analysis, namely, the so-called normal modes analysis. The nice feature of the theory is that appropriate boundary conditions are characterized by an algebraic condition known as the uniform Kreiss's condition. We illustrate this by presenting a formal program that determines whether boundary conditions are appropriate. Finally, examples of "good" and "bad" boundary conditions handled by the program are given.

Nomenclature

A	=	characteristic matrix
B	=	matrix of boundary conditions
c	=	sound speed of the fluid
D	=	reduced Lopatinskii determinant
d	=	dimension of the flow
I	=	identity matrix
i	=	square root of -1
L	=	reduced left eigenvector
ℓ	=	left eigenvector
$\boldsymbol{n}$	=	unit vector
P	=	polynomial of degree two
p	=	pressure
R	=	reduced right eigenvector
r	=	right eigenvector
S	=	symmetrizing matrix
T	=	temperature
t	=	time variable
u	=	tangential velocity
V	=	reduced frequency
v	=	normal velocity
$\boldsymbol{v}$	=	fluid velocity
$\boldsymbol{X}$	=	vector in the kernel of B
x	=	normal coordinate
$\boldsymbol{x}$	=	space variable
y	=	tangential coordinates
γ	=	adiabatic exponent
Δ	=	Lopatinskii determinant
$\boldsymbol{\zeta}$	=	orthogonal wave vector
$\boldsymbol{\eta}$	=	tangential wave vector
λ	=	eigenvalue
ξ	=	normal wave number
$\boldsymbol{\xi}$	=	wave vector
ρ	=	density
τ	=	complex frequency
Ω	=	reduced normal mode
ω	=	normal mode

Subscripts

$0, \pm$	=	numbering of (reduced) eigenmodes
1, 2, 3	=	numbering of eigenvalues and eigenvectors

Superscripts

t	=	transposed of a vector (or a matrix)
$\dot{}$	=	small perturbation of a constant state
$\hat{}$	=	Fourier transform (of a function)
$\bar{}$	=	constant value (of a function)
$\tilde{}$	=	shifted quantity by a real (or purely imaginary) number

I. Introduction

THE treatment of boundary conditions is known to be crucial in some areas of computational fluid dynamics (such as turbomachinery, etc.). With regard to the choice of appropriate boundary conditions, it remains quite mysterious and often relies only on repeated experiences. We focus here on Euler equations describing inviscid compressible fluid flows. A similar approach to clarify the choice of boundary conditions for Navier–Stokes equations would also be of interest, based on the work by Strikwerda,[1] for instance. However, this is beyond the scope of this paper. Examples of suitable boundary conditions and possible numerical treatments for Navier–Stokes equations can be found in Refs. 2–4.

With regard to Euler equations, it is widely acknowledged that the number of boundary data should coincide with the number of incoming characteristics (see Ref. 5, for instance). This number is determined by the nature of the boundary (inflow or outflow) and the expected flow itself (subsonic or supersonic). There is some indeterminacy for walls, which are "characteristic," but the appropriate number of boundary conditions can easily be understood by using geometrical arguments. However, the nature of boundary data has hardly been discussed, except in some works dating back to the 1970s,[4,6,7] where the "dissipativeness" of several kinds of data is examined. Dissipativeness refers to a method based on Friedrichs's symmetrization technique and "energy estimates" (see Refs. 8–10). Our purpose is to present a more systematic method that automatically predicts whether given boundary conditions are suitable. Here, suitable is a short cut for the mathematical well-posed nature (i.e., continuous dependence of the solution on the data) of the corresponding initial boundary value problem. This well-posed condition is a natural prerequisite before implementing a set of boundary conditions. We emphasize that numerical instabilities also may arise, even though the initial boundary value problem is theoretically well posed. This depends on the way of coding the boundary conditions, which is a more involved problem.[11,12] Here, we concentrate

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*Chargée de Recherche, Unité de Mathématiques Pures et Appliquées, Unité Mixte de Recherche 5669, École Normale Supérieure de Lyon, 46 Allée d'Italie.

†Student, Unité de Mathématiques Pures et Appliquées, Unité Mixte de Recherche 5669, École Normale Supérieure de Lyon, 46 Allée d'Italie.

‡Maître de Conférences, Département de Mécanique des Fluides et d'Acoustique, 36 Avenue Guy de Collongue.

on selecting boundary conditions that yield theoretical well-posed problems. The method we draw the attention to has been known for a long time in the theory of hyperbolic partial differential equations. It goes back to the seminal works by Hersh[13] and Kreiss,[14] and it is based on a normal modes analysis. However, despite its undoubted strength, it suffered because of its technicality, which explains why the energy method mentioned earlier has been preferred. Some attempts were made to popularize Kreiss's method (see, for instance, the review paper, Ref. 15). However, few pratical applications were reported. (The work announced in Ref. 4 by Oliger and Sundström never appeared.)

Surprisingly, a more complicated, but related, problem, namely, the stability of shock waves, was considered much more frequently. The derivation of a necessary (although not sufficient) stability condition by means of a normal modes analysis was performed by several authors a long time ago.[16–18] On this basis, the analysis was later refined by Majda[19–21] and Blokhin,[22,23] independently, who investigated whether the (generalized) Kreiss's condition is satisfied and proved the nonlinear stability of shocks that meet this condition. Stability of shocks is still a very active field in Blokhin's work, concerning shocks in more complicated hyperbolic systems: magnetohydrodynamics, relativistic dynamics, etc.

As far as we are concerned, there are basically three reasons why we have chosen to advocate Kreiss's method. The first reason is that the well-posed condition derived this way is known to be optimal. In particular, the energy method yields a more restrictive condition than Kreiss's one: A gap between these two conditions can be noticed in electromagnetism (see Ref. 24, p. 627, for instance). The second reason is that Kreiss's method is not as intricate as it may appear at first glance. We are going to dissect it in the context of Euler equations and show that its ingredients are just natural extensions to complex arguments of well-known ingredients (namely, eigenvalues and eigenvectors). Alternatively, the basic idea consists of looking at the standard dispersion relation the other way a round. The third reason is that technicality is no longer a problem with the help of symbolic calculation software. We provide here an adaptable tool, based on symbolic calculus, that reduces the verification of a well-posed condition to searching for zeros of a second-order polynomial (depending on two parameters, a frequency and a wave vector).

In Sec. II, we review the theoretical details of the normal modes analysis, first for noncharacteristic boundaries and then for slip-wall boundaries. The problem of domains with corners is briefly addressed. However, this case cannot be treated by the formal program. In Sec. III, we detail the operations of the program and show how to handle the results. Section IV is devoted to a series of examples that illustrate the different possible behaviors of boundary conditions.

II. Kreiss's Condition for Euler Equations

A. Background

Overall our analysis is concerned with the Euler equations for an ideal polytropic gas, linearized about a reference state $(\rho, \boldsymbol{v}, p)$. Notations are the standard ones: ρ is density, $\boldsymbol{v}$ velocity, and p static pressure. These are the most convenient variables for ideal polytropic gases, but other ones may be chosen if more complex models are preferred: One can choose, for instance, $(\rho, \boldsymbol{v}, e)$, with e being the specific internal energy, or $(p, \boldsymbol{v}, s)$, with s being the specific entropy. The reference state is intended to be the one achieved by the fluid at a fixed point of the boundary at a fixed time. We disregard what happens elsewhere. Linearization amounts to decomposing solutions of Euler equations into $(\rho + \dot\rho,\ \boldsymbol{v} + \dot{\boldsymbol{v}},\ p + \dot p)$ and retaining only the first-order terms with respect to the perturbation $(\dot\rho, \dot{\boldsymbol{v}}, \dot p)$. This yields the linear system of acoustics:

$$\partial_t\dot\rho + (\boldsymbol{v}\cdot\nabla)\dot\rho + \rho\nabla\cdot\dot{\boldsymbol{v}} = 0, \qquad \partial_t\dot{\boldsymbol{v}} + (\boldsymbol{v}\cdot\nabla)\dot{\boldsymbol{v}} + \rho^{-1}\nabla\dot p = 0$$

$$\partial_t\dot p + (\boldsymbol{v}\cdot\nabla)\dot p + \gamma p\nabla\cdot\dot{\boldsymbol{v}} = 0 \tag{1}$$

where γ is the adiabatic ratio of the gas. When performing a Fourier transform in $\boldsymbol{x}$ on system (1), we get the differential system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\hat\rho\\ \hat{\boldsymbol{v}}\\ \hat p\end{pmatrix} + iA(\boldsymbol{\xi})\begin{pmatrix}\hat\rho\\ \hat{\boldsymbol{v}}\\ \hat p\end{pmatrix} = 0$$

where the matrix $A(\boldsymbol{\xi})$, depending on wave vectors $\boldsymbol{\xi}\in\mathbb{R}^d$ (d is the dimension of the flow) reads

$$A(\boldsymbol{\xi}) \overset{\text{def}}{=} \left(\begin{array}{c|ccc|c} \boldsymbol{v}\cdot\boldsymbol{\xi} & & \rho\boldsymbol{\xi}^t & & 0\\ \hline & \boldsymbol{v}\cdot\boldsymbol{\xi} & & 0 & \\ 0 & & \ddots & & \dfrac{1}{\rho}\boldsymbol{\xi}\\ & 0 & & \boldsymbol{v}\cdot\boldsymbol{\xi} & \\ \hline 0 & & \gamma p\boldsymbol{\xi}^t & & \boldsymbol{v}\cdot\boldsymbol{\xi} \end{array}\right)$$

Important features of the matrix $A(\boldsymbol{\xi})$ are the following:

1) It has real eigenvalues, namely,

$$\lambda_1 \overset{\text{def}}{=} \boldsymbol{v}\cdot\boldsymbol{\xi} - c\|\boldsymbol{\xi}\|, \qquad \lambda_2 \overset{\text{def}}{=} \boldsymbol{v}\cdot\boldsymbol{\xi}, \qquad \lambda_3 \overset{\text{def}}{=} \boldsymbol{v}\cdot\boldsymbol{\xi} + c\|\boldsymbol{\xi}\|$$

where $c \overset{\text{def}}{=} \sqrt{(\gamma p/\rho)}$ is the sound speed,

2) It becomes symmetric when multiplied on the left by the symmetric positive-definite matrix

$$S \overset{\text{def}}{=} \left(\begin{array}{c|ccc|c} c^2 & & 0 & & -1\\ \hline 0 & \rho^2 & & 0 & 0\\ \vdots & & \ddots & & \vdots\\ 0 & 0 & & \rho^2 & 0\\ \hline -1 & & 0 & & 2c^{-2} \end{array}\right)$$

In particular, $A(\boldsymbol{\xi})$ is endowed with a complete set of eigenvectors. They read as follows. The eigenvalues λ_1 and λ_3 are simple, with left and right eigenvectors given by

$$\ell_j = \begin{bmatrix}0 & \rho(\lambda_j - \boldsymbol{v}\cdot\boldsymbol{\xi})\boldsymbol{\xi}^t & \|\boldsymbol{\xi}\|^2\end{bmatrix}, \qquad r_j = \begin{pmatrix}\rho\|\boldsymbol{\xi}\|^2\\ (\lambda_j - \boldsymbol{v}\cdot\boldsymbol{\xi})\boldsymbol{\xi}\\ \gamma p\|\boldsymbol{\xi}\|^2\end{pmatrix}$$

for $j = 1$ or 3. Here, we have intentionally kept the terms $(\lambda_j - \boldsymbol{v}\cdot\boldsymbol{\xi})$ in this form, even though they could be replaced by $\pm c\|\boldsymbol{\xi}\|$. This is to facilitate the subsequent extension procedure to complex values of $\boldsymbol{\xi}$.

The eigenvalue λ_2 is of multiplicity d. Independent families of left and right eigenvectors are given by

$$\ell_2^0 = (-c^2 \quad 0 \quad 1), \qquad r_2^0 = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\ell_2^k = [0 \quad (\boldsymbol{\zeta}^k)^t \quad 0], \qquad r_2^k = \begin{pmatrix}0\\ \boldsymbol{\zeta}^k\\ 0\end{pmatrix}$$

for $k = 1, \ldots, d-1$, where the vectors $\boldsymbol{\zeta}^k$ by definition span the $(d-1)$-dimensional vector space orthogonal to $\boldsymbol{\xi}$. For instance, if $d = 2$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$, we set

$$\boldsymbol{\zeta}_1 = \begin{pmatrix}-\xi_2\\ \xi_1\end{pmatrix}$$

When $d = 3$, the vectors $\boldsymbol{\zeta}_k$ are defined locally by similar formulas.

When dealing with a boundary value problem, we must consider two kinds of waves: those propagating across the boundary (the only ones occurring in one space dimension) and those propagating along the boundary. We shall refer to the former as normal waves and to the latter as tangential waves. In the sequence, we assume that the normal direction to the boundary is well defined. (This precludes points on edges and corners, which are much more complicated to deal with, see Sec. II.E.) To fix the ideas, we assume this normal direction to be the last coordinate axis. This is not a restriction, due to the isotropy of Euler equations.

Notations are as follows. For all positions $\boldsymbol{x}$ and velocities $\boldsymbol{v}$, we denote by x and v their normal components, respectively. Their tangential components will be denoted by y and u, respectively: that is, $y = x_1$ and $u = v_1$ in the case $d = 2$, and $y = (x_1, x_2)$ and $u = (v_1, v_2)$ in the case $d = 3$. Similarly, we will decompose wave vectors $\boldsymbol{\xi}$ into $(\boldsymbol{\eta}, \xi)$ with $\boldsymbol{\eta}$ a $(d-1)$-dimensional vector and ξ a scalar.

We choose the space orientation in such a way that the interior of the domain lies in $x > 0$. Solutions of interest of Eq. (1) in the subsequent normal modes analysis are those of the form

$$\begin{pmatrix} \dot\rho(t, y, x) \\ \dot{\boldsymbol{v}}(t, y, x) \\ \dot p(t, y, x) \end{pmatrix} = e^{\tau t} e^{i\eta\cdot y} e^{\omega x} \begin{pmatrix} \bar\rho \\ \bar{\boldsymbol{v}} \\ \bar p \end{pmatrix} \tag{2}$$

where τ is a complex frequency of positive real part, $\boldsymbol{\eta}$ is a real tangential wave vector, and ω is a complex wave number of non-positive real part. If such exploding modes (in time) are solutions of the initial boundary value problem, then this problem is clearly ill posed. Kreiss's analysis[14] is more subtle though. In particular, bounded modes of the form (2) with τ purely imaginary are also to be considered. We shall specify the concrete way of dealing with this case in Sec. II.B.

Observe that an algebraic condition for the interior system (1) to have solutions of the form (2) is that the matrix $\tau I + i A(\boldsymbol{\eta}, \omega/i)$ be singular, that is, noninvertible. Note that if this occurs for ω purely imaginary, then τ is necessarily purely imaginary because the eigenvalues of $A(\boldsymbol{\xi})$ are known to be real for $\boldsymbol{\xi} = (\boldsymbol{\eta}, \xi)$ real.

We are led to consider the characteristic matrix, not only for real vectors $\boldsymbol{\xi}$, but also for vectors $\boldsymbol{\xi}$, of which the last component ξ is complex. Let us go a little bit further. As a consequence of the preceding remark, the roots ω of the dispersion relation

$$\det[\tau I + i A(\boldsymbol{\eta}, \omega/i)] = 0 \tag{3}$$

are not allowed to cross the imaginary axis when $\mathrm{Re}\,\tau > 0$ and $\boldsymbol{\eta}$ is real. Indeed, if $\omega \in i\mathbb{R}$ satisfies Eq. (3) for $\mathrm{Re}\,\tau > 0$ and $\boldsymbol{\eta} \in \mathbb{R}^{d-1}$, then τ is an eigenvalue of the matrix $i A(\boldsymbol{\eta}, \omega/i)$, and we already know that this matrix has purely imaginary eigenvalues. This leads to a contradiction because $\mathrm{Re}\,\tau > 0$. This argument implies, in particular, that the number of roots ω of negative real part does not depend on $\boldsymbol{\eta}$ for $\mathrm{Re}\,\tau > 0$. And this number, for example, q, is easy to determine for $\boldsymbol{\eta} = 0$, that is, for the corresponding one-dimensional problem. As a matter of fact, Eq. (3) is equivalent for $\boldsymbol{\eta} = 0$ to

$$\tau = -\omega\lambda_j(0, 1)$$

for some $j = 1, 2$, or 3 [where the λ_j are the eigenvalues of the characteristic matrix $A(\boldsymbol{\xi})$, taken at $\boldsymbol{\xi} = (0, 1)$]. Therefore, q is nothing but the number of positive eigenvalues of $A(0, 1)$, that is, the number of incoming characteristics. Note that characteristics and the associated roots ω are counted with multiplicity (this means that λ_2 counts for d). These considerations are to be kept in mind in the following.

B. Normal Modes Analysis

In general, boundary data prescribe a certain number of nonlinear combinations of the primitive variables on the boundary. A first step in the theory of well-posed boundary conditions, as well as in practical implementation, consists of linearizing these boundary conditions about a reference state. A well-known prerequisite is that the number of such conditions be the number q of incoming characteristics (see Ref. 5, etc.). It will appear in the following that this is very reasonable, although not sufficient.

From now on, we consider a set of linearized boundary conditions in the form of a rectangular system

$$B \begin{pmatrix} \dot\rho \\ \dot{\boldsymbol{v}} \\ \dot p \end{pmatrix} = 0 \tag{4}$$

where B is a $q \times (d+2)$ matrix of rank q. A necessary condition for well-posedness is that there are no solutions to Eq. (1) of the form (2), other than zero, that meet the boundary conditions in Eq. (4). Because there are exactly q degrees of freedom in the solutions of the form (2) (according to the discussion in Sec. II.A), this amounts to requiring that some $(q \times q)$ determinant, denoted by $\Delta(\tau, \boldsymbol{\eta})$, does not vanish for $\mathrm{Re}\,\tau > 0$ and $\boldsymbol{\eta}$ real. Such a condition was first formalized by Hersh.[13] It was later pointed out by Kreiss[14] that it is not sufficient yet to ensure a well-posed problem. The suitable condition, known as the uniform Kreiss condition, requires that $\Delta(\tau, \boldsymbol{\eta})$ does not vanish even for τ purely imaginary [except at $(\tau, \boldsymbol{\eta}) = (0, 0)$]. Note that, in this criterion, the definition of $\Delta(\tau, \boldsymbol{\eta})$ for $\tau \in i\mathbb{R}$ is rather subtle. When $\tau \in i\mathbb{R}$, some of the roots ω of the dispersion relation (3) are purely imaginary and can not be selected according to the sign of their real parts. More precisely, the purely imaginary roots of interest in the definition of $\Delta(\tau, \boldsymbol{\eta})$ are those that are the continuous extension up to $\mathrm{Re}\,\tau = 0$ of roots ω that are of negative real part when $\mathrm{Re}\,\tau > 0$. This will be discussed in the sequel.

The subtlety of Kreiss's method mainly lies in two points: 1) considering the dispersion relation (3) for complex values of τ and ω with $\mathrm{Re}\,\tau > 0$ and afterward going back to purely imaginary values of τ (the usual case when considering characteristics) and 2) viewing the solutions of Eq. (3) as giving ω in terms of $(\tau, \boldsymbol{\eta})$ instead of τ in terms of $(\boldsymbol{\eta}, \omega)$. This makes a big difference, pointing out the existence of branching points on the set of solutions $(\tau, \boldsymbol{\eta}, \omega)$ of Eq. (3). When the solutions ω of Eq. (3) are seen as the eigenvalues of the matrix

$$-A_d^{-1}\left(\tau I + i\sum_{j=1}^{d-1} \boldsymbol{\eta}_j A_j\right)$$

those branching points correspond to values of $(\tau, \boldsymbol{\eta})$ such that the preceding matrix admits eigenvalues ω of algebraic multiplicity greater than their geometric multiplicity. The choice of appropriate branches in the computation of Δ is an important part of the subsequent job. We are going to explain in more detail how it works.

The dispersion relation (3) is easily seen to be equivalent to either

$$\tau + i\boldsymbol{\eta}\cdot u + v\omega = 0 \tag{5}$$

or

$$(\tau + i\boldsymbol{\eta}\cdot u + v\omega)^2 = c^2(\omega^2 - \|\boldsymbol{\eta}\|^2) \tag{6}$$

[Note that, replacing ω by $i\xi$ and τ by $-i\lambda$, we recover the characteristic relations $\lambda = \boldsymbol{v}\cdot\boldsymbol{\xi}$ or $(\lambda - \boldsymbol{v}\cdot\boldsymbol{\xi})^2 = c^2\|\boldsymbol{\xi}\|^2$.] For simplicity, we shall use the notation

$$\tilde\tau \stackrel{\mathrm{def}}{=} \tau + i\boldsymbol{\eta}\cdot u$$

(Note that $\mathrm{Re}\,\tilde\tau > 0$ is equivalent to $\mathrm{Re}\,\tau > 0$.) The linear equation (5) is very easy to deal with. Provided that v is nonzero, the only root is just $\omega = \omega_0$, where

$$\omega_0 \stackrel{\mathrm{def}}{=} -\tilde\tau/v \tag{7}$$

The second-order equation (6) is also rather easy to deal with when $\tilde\tau$ is not purely imaginary. In this case, we already know (from the discussion in Sec. II.A) that the solutions ω of Eq. (6) cannot be purely imaginary. Hence, the number of solutions of negative real part can be determined at $\boldsymbol{\eta} = 0$. However, for $\boldsymbol{\eta} = 0$, the solutions of Eq. (6) are explicitly given by

$$\omega = -[\tau/(v \pm c)]$$

In particular, we see that, in the subsonic case, that is, $|v| < c$, there is exactly one root of negative real part ω_-. However, for $\tilde\tau$ purely imaginary and $\tilde\tau^2 < (v^2 - c^2)\|\boldsymbol{\eta}\|^2$, Eq. (6) has two purely imaginary solutions, ω_- and ω_+. Which one should be retained for the definition of Δ? The answer is given by a complex analysis argument, namely, by the Cauchy–Riemann relations. Because the roots ω we are interested in are of negative real part when $\tilde\tau$ is of positive real part and are purely imaginary when $\tilde\tau$ is purely imaginary, their imaginary parts must be decreasing as the imaginary part of $\tilde\tau$ increases. This argument follows from the Cauchy–Riemann relation

$$\frac{\partial\,\mathrm{Re}\,\omega}{\partial\,\mathrm{Re}\,\tilde\tau} = \frac{\partial\,\mathrm{Im}\,\omega}{\partial\,\mathrm{Im}\,\tilde\tau}$$

This criterion provides a way to define univalued functions ω_- in terms of $(\tau, \boldsymbol{\eta})$ up to the imaginary axis (except at special points where, precisely, $\tilde{\tau}^2 = (v^2 - c^2)\,\|\boldsymbol{\eta}\|^2$; see Sec. III). Note that ω_- and ω_+ are not symmetric with respect to the imaginary axis.

C. Noncharacteristic Boundaries and Nonsonic Flows

There are four intrinsic cases, which can be ordered by increasing difficulty. This classification is, of course, independent of our choice of the space orientation, even though we use this choice to fix the ideas.

According to the domain $\{x > 0\}$, inflows and outflows are selected through the sign of v as follows. Inflow corresponds to $v > 0$ and outflow to $v < 0$.

1. Supersonic Outflow

All characteristics are outgoing. As a matter of fact, supersonic outflow corresponds with our choice to $-v > c > 0$. In this case, all eigenvalues of $A(0, 1)$ are negative, that is, all characteristics are outgoing of the domain $\{x > 0\}$. No boundary condition is required. There is nothing to check. We do not claim that numerical treatment is easy, but it must not rely on any boundary data.

2. Supersonic Inflow

All characteristics are incoming. This is because $v > c > 0$ implies that all eigenvalues of $A(0, 1)$ are positive. Hence, the boundary matrix B must be a square invertible matrix. The invertibility of B is the only thing to check. It should be enough to make the numerical treatment work (entirely relying on boundary data).

3. Subsonic Outflow

Only one characteristic is incoming. This is because, for $0 < -v < c$, the only positive eigenvalue of $A(0, 1)$ is $\lambda_3(0, 1)$. Thus, there is only one boundary condition to impose. In other words, B must be a (nonzero) row matrix. It must not cancel the eigenmode corresponding to the unique solution ω_- of Eq. (6) of negative real part. This eigenmode is necessarily proportional to the right eigenvector of $A(\boldsymbol{\eta}, \omega_-/i)$, defined by

$$r_-(\tau, \boldsymbol{\eta}) \stackrel{\text{def}}{=} r_3(\boldsymbol{\eta}, \omega_-/i) = \begin{bmatrix} \rho\left(\|\boldsymbol{\eta}\|^2 - \omega_-^2\right) \\ (i\tau - u\cdot\boldsymbol{\eta} + iv\omega_-)\boldsymbol{\eta} \\ (\tau + iu\cdot\boldsymbol{\eta} + v\omega_-)\omega_- \\ \gamma p\left(\|\boldsymbol{\eta}\|^2 - \omega_-^2\right) \end{bmatrix}$$

Using Eq. (6), we have

$$r_-(\tau, \boldsymbol{\eta}) \,/\!/ \begin{bmatrix} -\rho(\tilde{\tau} + v\omega_-)/c^2 \\ i\boldsymbol{\eta} \\ \omega_- \\ -\rho(\tilde{\tau} + v\omega_-) \end{bmatrix} \tag{8}$$

The "determinant" Δ is just obtained by taking the product of B and r_-.

4. Subsonic Inflow

There are $(d + 1)$ incoming characteristics (counting with multiplicity). This is the more complicated case, where $0 < v < c$ and, thus, $q = d + 1$. (Recall that λ_2 counts for d.) The eigenmodes of the form (2) to be taken into account are those corresponding to the unique solution ω_- of Eq. (6) of negative real part and those corresponding to ω_0. Specifically, B must not vanish on the $(d+1)$-dimensional subspace spanned by $r_3(\boldsymbol{\eta}, \omega_-/i)$ and $r_2^k(\boldsymbol{\eta}, \omega_0/i)$, $k = 0, \ldots, d-1$. Note that this subspace is not the orthogonal subspace to some vector ℓ_1 because the arguments in r_3 and r_2^k are different. However, we can remark that

$$\operatorname{span}\left\{r_2^0(\boldsymbol{\eta}, \omega_0/i), \ldots, r_2^{d-1}(\boldsymbol{\eta}, \omega_0/i), r_3(\boldsymbol{\eta}, \omega_-/i)\right\} = \ell_-(\tau, \boldsymbol{\eta})^\perp$$

where

$$\ell_-(\tau, \boldsymbol{\eta}) \stackrel{\text{def}}{=} \begin{bmatrix} 0 & i\boldsymbol{\eta}^t & \omega_0 & \dfrac{\omega_0\omega_- - \|\boldsymbol{\eta}\|^2}{\rho(\tilde{\tau} + v\omega_-)} \end{bmatrix}$$

Using Eqs. (6) and (7), we note that ℓ_- also reads

$$\ell_-(\tau, \boldsymbol{\eta}) = \begin{pmatrix} 0 & i\boldsymbol{\eta}^t & -\dfrac{\tilde{\tau}}{v} & \dfrac{\tilde{\tau} + v\omega_-}{\rho c^2} - \dfrac{\omega_-}{\rho v} \end{pmatrix}$$

The main feature of ℓ_- that will be useful is that ω_- appears linearly. Once we have ℓ_-, an easy (although not optimal) way to define Δ is to consider the $(d + 2) \times (d + 2)$ determinant

$$\Delta(\tau, \boldsymbol{\eta}) = \begin{vmatrix} B \\ \ell_-(\tau, \boldsymbol{\eta}) \end{vmatrix}$$

An alternative definition of $\Delta(\tau, \boldsymbol{\eta})$ is

$$\Delta(\tau, \boldsymbol{\eta}) = \left| Br_2^0(\boldsymbol{\eta}, \omega_0/i) \quad \ldots \quad Br_2^{d-1}(\boldsymbol{\eta}, \omega_0/i) \quad Br_3(\boldsymbol{\eta}, \omega_-/i) \right|$$

However, though this latter expression is only a $(d + 1) \times (d + 1)$ determinant, it involves more complicated expressions than the preceding one, and we have preferred the first definition of Δ to treat this case.

D. Slip Walls

The case of slip walls falls into characteristic boundaries. From a theoretical point of view, this is a more difficult case, which was first addressed in some generality by Majda and Osher.[24] Anterior works (for instance, Refs. 9 and 10) had dealt with examples of characteristic boundaries, motivated by physical examples, by the energy method.

Assume that the boundary $\{x = 0\}$ is a rigid slip wall or, in other words, that the normal component of the velocity vanishes at the boundary. This physical constraint reads, in our notations, $v = 0$, and naturally yields one linear(ized) boundary condition

$$\dot{v} = 0 \tag{9}$$

Following Majda and Osher's[24] approach, we are going to see that no other condition should be specified on the boundary.

First, we note that, because $v = 0$, all vectors of the form

$$\begin{pmatrix} \dot{\rho} \\ \dot{u} \\ 0 \\ 0 \end{pmatrix}$$

belong to the kernel of the normal matrix $A(0, 1)$. Furthermore, according to Ref. 24, it is necessary that B annihilates such vectors to ensure a well-posed condition. (Thus, B can be at most of rank 2.)

Furthermore, the only eigenmode to be considered is associated with ω_-, the unique solution of Eq. (6) of negative real part. Hence, B must be a row matrix. Logically, in view of Eq. (9), we just take $B = (0, 0, 1, 0)$.

Therefore, we simply define

$$\Delta(\tau, \boldsymbol{\eta}) = \omega_-(\tau, \boldsymbol{\eta})$$

[applying B to the right-hand side in Eq. (8)].

Thus, the question is whether ω_- vanishes. This is obviously not the case for Re $\tau > 0$. However, we clearly have $\omega_-(\tau, \boldsymbol{\eta}) = 0$ for $\tilde{\tau} = \pm ic\|\boldsymbol{\eta}\|$. In other words, the bounded modes

$$\exp[i(\pm c\|\boldsymbol{\eta}\|t + \boldsymbol{\eta}\cdot y)] \begin{pmatrix} \pm i\rho|\boldsymbol{\eta}|/c \\ -i\boldsymbol{\eta} \\ 0 \\ \pm i\rho c|\boldsymbol{\eta}| \end{pmatrix}$$

form boundary waves propagating at speed c: They are solutions of Eq. (1) of the form (2) and satisfy the boundary condition (9).

Although the (uniform) Kreiss's condition fails in this case, it has been shown in Ref. 25 that the initial boundary value problem is well posed in a weaker sense. However, special attention should be paid to the numerical treatment of this boundary condition because the failure of a uniform well-posed condition may give rise to numerical instabilities. Different boundary schemes can be found in Refs. 6

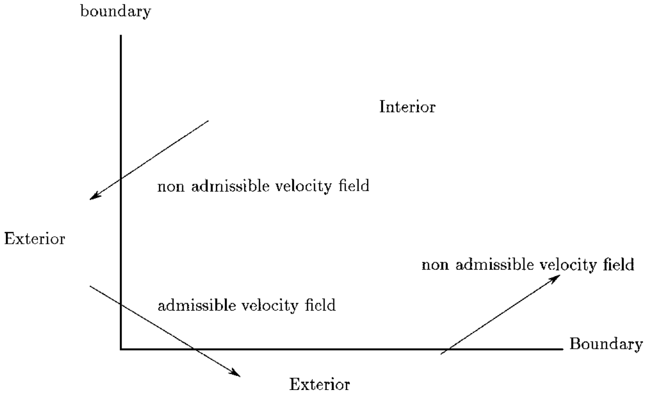

Fig. 1 Admissible/nonadmissible velocities in space dimension two for Osher's theory.[29,30]

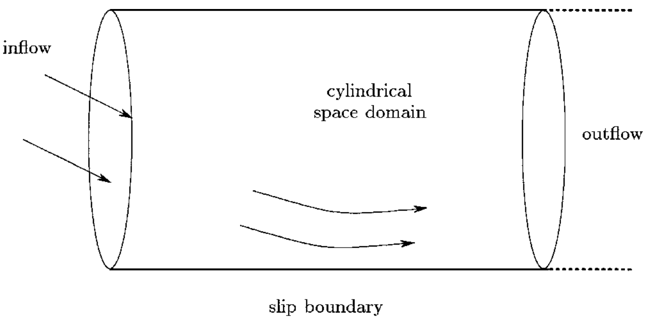

Fig. 2 Nonadmissible domain in space dimension three.

and 26. Even though stability results have been proved for one-dimensional problems in several cases,[7,27,28] convergence remains empirical. (See Refs. 2, 3, and 6 for a nice discussion of this topic.) Possible numerical treatments are proposed in Ref. 26 for multidimensional problems. (References for other types of problems, such as reactive flows or magnetohydrodynamics, can be found therein.)

E. Edges and Corners

Problems due to lack of smoothness of the boundary were addressed by Osher.[29,30] He pointed out that a well-posed condition on each side of a wedge does not ensure a well-posed nature of the overall problem. Examples of undesirable phenomena in dimension two (nonuniqueness, singularities) are reported in Ref. 30. However, Osher's work[29,30] is a tough piece of mathematical analysis. It is hardly applicable as a "black box." Furthermore, it suffers from at least two drawbacks:

1) It cannot deal with all possible mean flows. This is specified in Fig. 1. (Mean flow velocities are called "nonadmissible" when Osher's work[29,30] cannot apply; other velocities are called "admissible.")

2) It does not include the physical case of characteristic boundaries, for instance, the motion of a fluid in a tube with slip boundaries (Fig. 2). This common situation in fluid dynamics gathers almost all difficulties encountered in the study of mixed problems. We refer the reader to Ref. 31 for recent results on this problem.

In a more general setting, the analysis of initial boundary value problems is much more complicated than the analysis of initial value problems (also called Cauchy problems), partly because the space–time domain of study presents a corner. When the space domain also presents a corner, many compatibility conditions are required, and this is beyond the scope of this paper.

III. Systematic Testing

A. General Framework

Ready-made routines in Maple are available at URL: http://www.umpa.ens-lyon.fr/~jfcoulom/. There, the user can choose the space dimension ($d = 1, 2,$ or 3), the type of boundary (inflow or outflow), and the regime of the fluid (subsonic or supersonic). Depending on these choices, the appropriate number of boundary conditions is returned. Then boundary conditions can be entered as nonlinear combinations of the variables $(\rho, \boldsymbol{v}, p)$. They are subsequently linearized by symbolic calculation. The program can not deal with absorbing boundary conditions. For recent results on this alternative approach, we refer the reader to Refs. 32 and 33, for instance.

Recalling the four cases from Sec. II.C, we see that the most complicated testings occur for subsonic flows. For supersonic inflows, symbolic calculation may just help to check that the chosen boundary conditions are independent, that is, that B is invertible. From now on, we concentrate on subsonic flows.

B. Handling the Output for Subsonic Flows

One-dimensional problems are almost straightforward to decide, relying on just one point testing. For $\boldsymbol{\eta} = 0$, ω_0, and ω_- are homogeneous of degree one in τ. Hence, it is sufficient to check that $\Delta(1, 0)$ is nonzero to ensure that the uniform Kreiss's condition is fulfilled. Note that otherwise Kreiss's condition merely fails.

For $\boldsymbol{\eta} \neq 0$, one can reduce the problem by using the new quantities

$$V := \tau / i\|\boldsymbol{\eta}\|, \qquad \boldsymbol{n} := \boldsymbol{\eta}/\|\boldsymbol{\eta}\|, \qquad \Omega_0 := \omega_0 / i\|\boldsymbol{\eta}\|$$
$$\Omega_- := \omega_- / i\|\boldsymbol{\eta}\|$$

The first one is homogeneous to a velocity and the others are nondimensional. Observe that V has nonpositive imaginary part and that Ω_0 and Ω_- have nonnegative imaginary part in the range considered. For $d = 2$, we just let $\boldsymbol{n} = 1$ (noting that the case $\boldsymbol{n} = -1$ is obtained by conjugation). For three-dimensional problems, $\boldsymbol{n}$ is a unit vector in $\mathbb{R}^2$, which we parametrize by an angle θ.

In view of Eqs. (6) and (7) we have

$$\Omega_0 = -(V + \boldsymbol{n} \cdot u)/v \tag{10}$$

$$(V + \boldsymbol{n} \cdot u + v\,\Omega_-)^2 = c^2\left(\Omega_-^2 + 1\right) \tag{11}$$

with $\operatorname{Im}\,\Omega_- > 0$ for $\operatorname{Im}\,V < 0$. If V is real, that is, if τ is purely imaginary, and $\tilde{V}^2 \geq (c^2 - v^2)$, the expression of Ω_- is explicitly given by the argument given in Sec. II.B. We have

$$\Omega_- = \frac{v\tilde{V} - \operatorname{sgn}(V)c\sqrt{\tilde{V}^2 - (c^2 - v^2)}}{c^2 - v^2} \tag{12}$$

with $\tilde{V} = V + \boldsymbol{n} \cdot u$. Otherwise, Ω_- still has a positive imaginary part and is given by

$$\Omega_- = \frac{v\tilde{V} + ic\sqrt{(c^2 - v^2) - \tilde{V}^2}}{c^2 - v^2} \tag{13}$$

We can use the following vectors to define a reduced determinant $D(V)$ proportional to $\Delta(\tau, \boldsymbol{\eta})$. Defining

$$R_-(V) \overset{\text{def}}{=} \begin{pmatrix} (\tilde{V} + v\,\Omega_-)/c^2 \\ -\boldsymbol{n}/\rho \\ -\Omega_-/\rho \\ \tilde{V} + v\Omega_- \end{pmatrix}$$

$$L_-(V) \overset{\text{def}}{=} \left(0 \quad \boldsymbol{n}^t \quad -\frac{\tilde{V}}{v} \quad \frac{\tilde{V} + v\Omega_-}{\rho c^2} - \frac{\Omega_-}{\rho v}\right)$$

we see that

$$R_-(V) \,/\!/\, r_-(iV\|\boldsymbol{\eta}\|, \boldsymbol{\eta}), \qquad L_-(V) \,/\!/\, \ell_-(iV\|\boldsymbol{\eta}\|, \boldsymbol{\eta})$$

with r_- and ℓ_- defined as in Sec. II.C. Therefore, suitable definitions for $D(V) = \Delta(iV\|\boldsymbol{\eta}\|, \boldsymbol{\eta})$ are either

$$D(V) \overset{\text{def}}{=} BR_-(V)$$

for subsonic outflows, or

$$D(V) \overset{\text{def}}{=} \begin{vmatrix} B \\ L_-(V) \end{vmatrix}$$

for subsonic inflows. In both cases, because Ω_- appears linearly in R_- and L_-, we observe that $D(V) = 0$ amounts to a linear relation in Ω_- of the form

$$a(V)\,\Omega_- = b(V)$$

with the additional properties that $a(V)$ and $b(V)$ are affine in $\tilde{V}$, with real coefficients.

A special case arises if a and b vanish simultaneously. (An example is given in Sec. IV.) We emphasize that this is the only case when the uniform Kreiss condition may fail with Ω_- being nonreal. Otherwise, if $a(V) \neq 0$ then $\Omega_- = b(V)/a(V)$ is necessarily real for V real. In other words, the case $a(V) = b(V) = 0$ is the only one for which surface waves, that is, oscillating modes of the form (2) localized near the boundary (with Re $\omega < 0$ and τ purely imaginary) may occur. The mathematical study of such problems where surface waves occur is not yet much developed. Some results may be found in Ref. 34.

Let us assume now that $a(V) \neq 0$ and define

$$\Omega_*(V) \stackrel{\text{def}}{=} b(V)/a(V)$$

Substituting $\Omega_- = \Omega_*(V)$ in Eq. (11) yields a second-order equation for V, for example, $P(V) = 0$. Observe that the (at most) second-order polynomial P has real coefficients that can be derived by an appropriate symbolic calculation routine. Now, several cases may arise that can be tested numerically.

1) Either of the roots of P are nonreal. Then, one and only one of them, for example, V_0, is of negative imaginary part and, thus, is a candidate for unstable mode. If $\Omega_*(V_0)$ is of positive imaginary part, then it coincides with $\Omega_-(V_0)$, defined by Eq. (11), and Im $\Omega_- > 0$. Thus, we do have $D(V_0) = 0$. This means that Kreiss's condition fails at point V_0. Otherwise the uniform Kreiss condition is satisfied. (D cannot vanish for real V.)

2) Another case is where P has real roots, for example, V_1 and V_2. Then D does not vanish for Im $V < 0$. This means that Kreiss's condition is satisfied. We need to check whether the uniform Kreiss condition holds. For $\Omega_-(V_j)$ to be real, we must have $(V_j + \boldsymbol{n}\cdot u)^2 \geq c^2 - v^2$. If this condition fails for both $j = 1$ and 2, then the determinant D does not vanish for V real. Thus, the uniform Kreiss condition is satisfied. Otherwise, if $(V_j + \boldsymbol{n}\cdot u)^2 \geq c^2 - v^2$, we must determine whether $\Omega_-(V_j)$ [given by Eq. (12)] is equal to $\Omega_*(V_j)$. It can happen that $\Omega_*(V_j)$ is the other root of Eq. (11). If $\Omega_*(V_j) = \Omega_-(V_j)$, there exist boundary waves of speed V_j. If not, the uniform Kreiss condition is satisfied.

These arguments are summarized in Fig. 3.

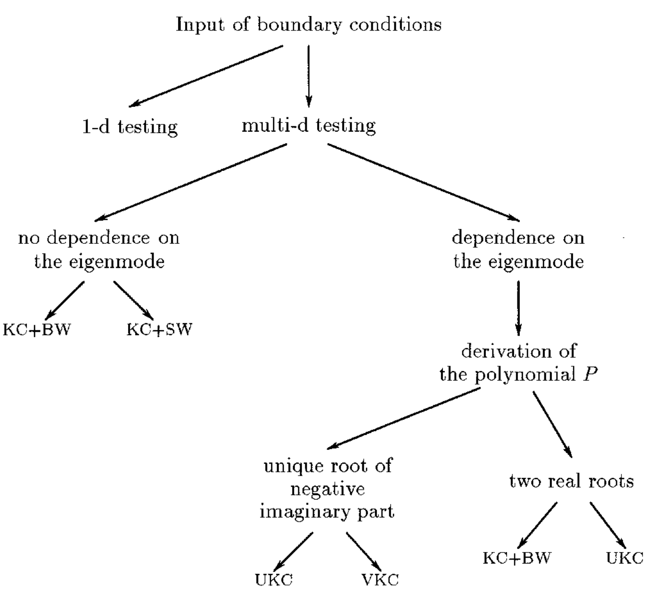

Fig. 3 Testing progress: KC, Kreiss's condition; BW, boundary waves; SW, surface waves; UKC, uniform Kreiss's condition; and VKC, violated Kreiss's condition.

For three-dimensional problems, the computations cannot always be carried to their end because of the dependence on the angle θ. A critical situation appears when the discriminant of P, which depends only on θ once numerical values are given, may change sign. In such a case, one has to distinguish between different cases. If the calculations turn out to be too cumbersome, one can always check the energy dissipation of boundary conditions. By definition, this requires that the symmetric matrix

$$SA(0,1) = \begin{pmatrix} vc^2 & 0 & 0 & 0 & -v \\ 0 & \rho^2 v & 0 & 0 & 0 \\ 0 & 0 & \rho^2 v & 0 & 0 \\ 0 & 0 & 0 & \rho^2 v & \rho \\ -v & 0 & 0 & \rho & 2vc^{-2} \end{pmatrix}$$

be negative when restricted to the kernel of B, namely,

$$\forall \boldsymbol{X} \in \ker B, \boldsymbol{X} \neq 0, \qquad \boldsymbol{X}^t SA(0,1)\boldsymbol{X} < 0$$

This is sometimes easier to check and implies the uniform Kreiss condition, as was already mentioned (see Ref. 35, Chapter 14).

IV. Examples of Boundary Conditions

We keep focusing on subsonic flows.

A. Out of Question Boundary Data

1. Subsonic Outflows

Looking at $r_-(1,0)$, we see that choosing a boundary condition only in terms of the tangential velocity would yield an ill-posed condition.

2. Subsonic Inflows

Because the first component of $\ell_-(\tau, \boldsymbol{\eta})$ is zero, if the first column of B is also zero, then $\Delta(\tau, \boldsymbol{\eta}) = 0$ for all τ and $\boldsymbol{\eta}$. Kreiss's condition fails even in dimension one. This means that prescribing velocity and pressure leads to a violently ill-posed problem. (This had already been pointed out in Ref. 12.) The density of the fluid must be involved in the quantities prescribed on the entry.

B. Good Boundary Data in Dimension One

For subsonic outflows, one may prescribe, indifferently, density, velocity, or pressure.

For a subsonic inflow, here are two examples that fully work in dimension one only. One may prescribe either of the following:

1) Pressure and temperature may be prescribed, which give

$$B = \begin{pmatrix} 0 & 0 & 1 \\ -p/\rho^2 & 0 & 1/\rho \end{pmatrix}$$

2) Velocity and temperature may be prescribed instead, which give

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -p/\rho^2 & 0 & 1/\rho \end{pmatrix}$$

C. Poor Boundary Data in Dimension Two

Let us try to extend the two preceding examples for subsonic inflows to dimension two.

For instance, one may prescribe tangential velocity, pressure, and temperature, which correspond to

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -p/\rho^2 & 0 & 0 & 1/\rho \end{pmatrix}$$

We find that $D(V)$ is proportional to $(V + u)$, that is, $\tilde{V}$, which does not vanish if V has a negative imaginary part. Kreiss's condition is, therefore, fulfilled. It is also obvious that this determinant vanishes for $V = -u$, that is, for $\tilde{V} = 0$. The corresponding reduced eigenmodes, of nonnegative imaginary part, are given by Eqs. (10) and (13) and have values

$$\Omega_0 = 0, \qquad \Omega_- = ic\Big/\sqrt{c^2 - v^2}$$

This corresponds to boundary waves propagating at speed $|u|$.

One may attempt to prescribe the whole velocity and temperature, thus generalizing example 2. This means taking

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -p/\rho^2 & 0 & 0 & 1/\rho \end{pmatrix}$$

With the notations introduced in Sec. III, we find that

$$a(V) = V + u, \qquad b(V) = v$$

which yields

$$P(V) = (V + u)^2 + v^2 - c^2$$

We see that P has real roots. We are in the case when Kreiss's condition is satisfied but there are boundary waves propagating at speed $|-u \pm \sqrt{(c^2 - v^2)}|$.

D. Good Boundary Data in Dimension Two

For a subsonic outflow, prescribing the pressure works perfectly well. With the notations of Sec. III, we find that

$$a(V) = -v, \qquad b(V) = V + u$$

$$P(V) = V^2 + 2uV + u^2 + v^2 = 0$$

The roots of P are nonreal and Im $\Omega_*(V) < 0$. Therefore, the uniform Kreiss condition is satisfied.

The same result holds if density is preferred to pressure.

E. Dissipative Boundary Data in Dimension Three

We consider a subsonic inflow where the quantities prescribed at the boundary are the total pressure, the total temperature, and the flow angles:

$$p\{1+[(\gamma-1)/2](\|\boldsymbol{v}\|^2/c^2)\}^{\gamma/(\gamma-1)}, \qquad T+\|\boldsymbol{v}\|^2\big/2c_p, \qquad \boldsymbol{v}/\|\boldsymbol{v}\|$$

where T is temperature and c_p is the heat capacity per unit of mass. Once linearized, these boundary conditions yield a matrix B, of which the kernel is the one-dimensional subspace spanned by the vector

$$\boldsymbol{X}_0 \stackrel{\text{def}}{=} \begin{pmatrix} -\rho\|\boldsymbol{v}\|^2/c^2 \\ \boldsymbol{v} \\ -\rho\|\boldsymbol{v}\|^2 \end{pmatrix}$$

An easy computation shows that

$$\boldsymbol{X}_0^t SA(0, 1)\boldsymbol{X}_0 = -\rho^2 v\|\boldsymbol{v}\|^2(1 - \|\boldsymbol{v}\|^2/c^2) < 0$$

This automatically implies the uniform Kreiss condition.

For subsonic outflows, it is obvious that pressure is a dissipative boundary condition because the upper left $(d+1) \times (d+1)$ block in $SA(0, 1)$ is diagonal with negative coefficients $(v < 0)$.

V. Conclusions

The problem of boundary conditions for acoustic systems has been addressed with a new approach: The formal program described in this paper makes it possible to check a uniform well-posed condition for arbitrary boundary conditions once the regime of the fluid (dimension of the flow, subsonic or supersonic flow) has been chosen. This program gives a selection criterion of boundary conditions that may be useful to scientists or engineers willing to simulate unsteady flows in domains with a more or less complex geometry.

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P. Givi
Associate Editor